On Strong Tree-Breadth

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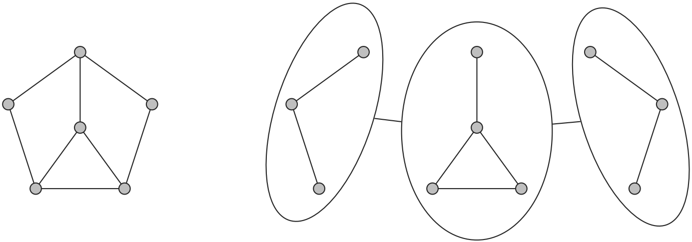
Introduction and Motivation

Tree-Decomposition of a Graph

A family $\mathcal{T} = \{B_1, B_2, \ldots, B_k\}$ of subsets of V (called bags) which form a tree such that

- each vertex is in a bag,
- each edge is in a bag, and
- the bags containing a vertex induce a subtree.



For a family $\mathcal{T}$, it can be checked in linear time if it is a tree-decomposition.

New(-ish) Concept: Tree-Breadth

Breadth of a Decomposition

- Maximum radius ρ of all bags B

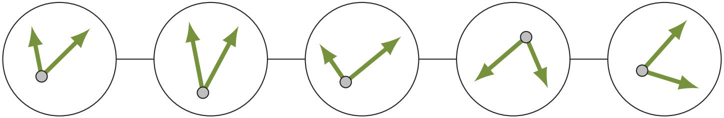
$$\text{breadth}(\mathcal{T}) = \min\left\{ \rho \;\middle|\; \forall B \in \mathcal{T} \;\exists v \in V\colon B \subseteq N^{\rho}[v] \right\}$$

- Gives a center v for each bag.

Tree-Breadth of a Graph

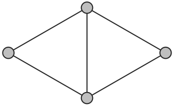
- Smallest breadth of all tree-decompositions $\mathcal{T}$ for G

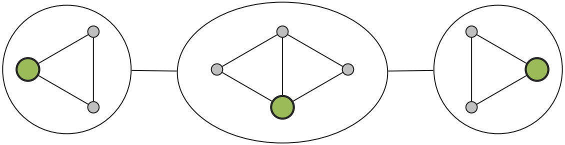
$$\text{tb}(G) = \min\{ \rho \mid \forall \mathcal{T}\colon \text{breadth}(\mathcal{T}) \le \rho \}$$



Dually Chordal Graphs

A graph $G = (V, E)$ with $V = \{v_1, v_2, \ldots, v_n\}$ is *dually chordal* if $\mathcal{N} = \{N[v_1], N[v_2], \ldots, N[v_n]\}$ is a tree-decomposition for G.





$2\times$

Dually Chordal vs Tree-Breadth

Dually Chordal
$B = N[v]$ for *all* v

Dually Chordal vs Tree-Breadth

Dually Chordal		Tree-Breadth
$B = N[v]$ for *all* v	$\subseteq$	$B \subseteq N[v]$ for *some* v

Dually Chordal vs Tree-Breadth

$B \subseteq N[v]$ for all v

$\subsetneq$ $\quad\quad\quad$ $\subsetneq$

Dually Chordal
$B = N[v]$ for *all* v

$\subseteq$

Tree-Breadth
$B \subseteq N[v]$ for *some* v

Dually Chordal vs Tree-Breadth

$B \subseteq N[v]$ for all v

$\subseteq$ $=$

Dually Chordal
$B = N[v]$ for *all* v

$\subseteq$

Tree-Breadth
$B \subseteq N[v]$ for *some* v

$\supseteq$ $\subseteq$

$B = N[v]$ for some v

Dually Chordal vs Tree-Breadth

$B \subseteq N[v]$ for all v

$\subseteq$ $=$

Dually Chordal
$B = N[v]$ for *all* v

$\subseteq$

Tree-Breadth
$B \subseteq N[v]$ for *some* v

$\supseteq$ $\subseteq$

$B = N[v]$ for some v
Strong Tree-Breadth

$$\text{strongBreadth}(\mathcal{T}) = \min \left\{ \rho \;\middle|\; \forall B \in \mathcal{T} \; \exists v \in V \colon B = N^{\rho}[v] \right\}$$

$$\text{stb}(G) = \min \left\{ \rho \;\middle|\; \forall \mathcal{T} \colon \text{strongBreadth}(\mathcal{T}) \leq \rho \right\}$$

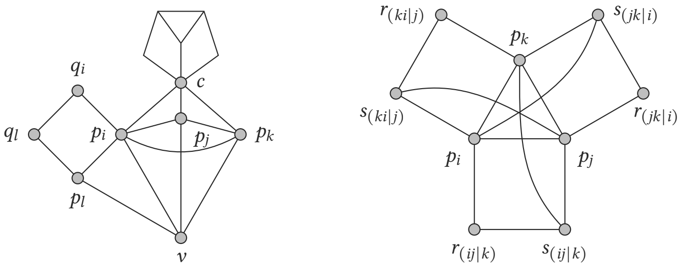
Results

Determining Strong Tree-Breadth of a Graph

Theorem

For a given graph G and a given integer ρ, it is NP-complete to determine if $\mathrm{stb}(G) \leq \rho$, even for $\rho = 1$.

Reduction from 1-in-3-SAT



Subgraphs of G as created by a clause $c = \{p_i, p_j, p_k\}$ and a literal p_l with $p_i \equiv \neg p_l$.

Computing a Decomposition

Theorem

For a graph G with $\mathrm{stb}(G) \leq \rho$, a tree-decomposition $\mathcal{T}$ with "weak" breadth ρ can be computed in polynomial time.

Perfect Strong Tree-Breadth

- For two adjacent bags $N[u]$ and $N[v]$, $N[u]$ intersect only one connected component of $G - N[v]$.

Theorem

If a graph admits a tree-decomposition with perfect strong breadth ρ, such a decomposition can be constructed in polynomial time.

Observation

- If, in a decomposition $\mathcal{T}$ with strong breadth ρ, the distances of centers are at least ρ, then $\mathcal{T}$ has perfect strong breadth ρ.

Strong Tree-Breadth for Special Graph Classes

Theorem

Distance-hereditary graphs, chordal graphs, chordal bipartite graphs, and permutation graphs have strong tree-breadth 1.

For all these classes, an according tree-decomposition can be computed in linear time.

Open Questions

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Strong *Path*-Breadth

- How hard is it to determine the strong path-breadth of a graph?
- Conjecture: Doable in polynomial time.

Difference to “weak” Tree-Breadth

- Is there a constant c such that, for all graphs G, $\mathrm{stb}(G) \leq c \cdot \mathrm{tb}(G)$?
- Conjecture: 2, 3, or *none*.

Thank You!